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INTEGER AND FRACTIONAL PACKINGS IN DENSE GRAPHS

P. E. HAXELL, V. RÖDL

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Let H_0 be any fixed graph. For a graph G we define $\nu_{H_0}(G)$ to be the maximum size of a set of pairwise edge-disjoint copies of H_0 in G. We say a function ψ from the set of copies of H_0 in G to $[0,1]$ is a *fractional H_0-packing* of G if $\sum_{H\ni e}\psi(H)\le 1$ for every edge e of G. Then $\nu^*_{H_0}(G)$ is defined to be the maximum value of $\sum_{H\in\binom{G}{H_0}}\psi(H)$ over all fractional H_0-packings ψ of G. We show that $\nu^*_{H_0}(G)-\nu_{H_0}(G)=o(|V(G)|^2)$ for all graphs G.

1. Introduction

In this note we study a problem of packing edge-disjoint copies of a fixed graph H_0 into a given graph G. In order to state our main result we first give some notation and definitions. An H_0-*packing* in G is defined to be a set of copies of H_0 in G which are pairwise edge-disjoint. We denote the set of copies of H_0 in G by $\binom{G}{H_0}$, and we write $\nu_{H_0}(G)$ for the maximum size of an H_0-packing in G. For a graph G, a *fractional H_0-packing* of G is a function $\psi:\binom{G}{H_0}\to[0,1]$ which satisfies $\sum_{H\ni e}\psi(H)\le 1$ for every $e\in E(G)$. We let $|\psi|=\sum_{H\in\binom{G}{H_0}}\psi(H)$, and we say that ψ is a *maximum* fractional H_0-packing of G if $|\psi|$ is as large as possible. We denote by $\nu^*_{H_0}(G)$ the value $|\psi|$ of a maximum fractional H_0-packing ψ of G. Clearly $\nu_{H_0}(G)\le\nu^*_{H_0}(G)$ for any graph G.

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Our main theorem is the following.

Theorem 1. *Let H_0 be a fixed graph, and let $\eta > 0$ be given. Then there exists $N = N(\eta, H_0)$ such that for any graph G with $n \geq N$ vertices we have*

$$\nu^*_{H_0}(G) - \nu_{H_0}(G) \leq \eta n^2.$$

Moreover an H_0-packing in G of size at least $\nu^*_{H_0}(G) - \eta n^2$ can be found in time polynomial in n.

For any graph H_0 that contains a connected component with at least three edges, the problem of finding $\nu_{H_0}(G)$ for a general graph G is NP-hard (see Dor and Tarsi [5]). Since finding $\nu^*_{H_0}(G)$ is a linear programming problem and hence solvable in polynomial time, Theorem 1 gives an efficient algorithm for approximating $\nu_{H_0}(G)$ for those graphs G for which $\nu_{H_0}(G) = \Omega(|V(G)|^2)$. Therefore this problem is another example of an NP-hard problem which has a polynomial time approximation algorithm for an appropriately defined "dense case". Various other such problems have been identified and studied by *e.g.* Frieze and Kannan [8], [9] and Arora, Karger and Karpinski [3] (see also [6] and [4]). In many of these problems, as in ours, the Regularity Lemma of Szemerédi plays an important role. As well as finding an approximation to $\nu_{H_0}(G)$, our result also gives an algorithm for finding an H_0-packing of nearly optimal size.

We prove Theorem 1 by showing that there exists a finite weighted graph G_0, *i.e.* whose size is independent of n, such that both $\nu_{H_0}(G)$ and $\nu^*_{H_0}(G)$ are closely approximated by $\nu^*_{H_0}(G_0)n^2/s^2$, where $s = |V(G_0)|$ (see Theorem 9). This graph is obtained by an application of the Regularity Lemma of Szemerédi. In addition to Szemerédi's lemma [13], and an algorithmic version of it due to Alon, Duke, Lefmann, Rödl and Yuster [1], the proof of Theorem 1 depends on a packing result of Frankl and Rödl [7] and an algorithmic version of this by Grable [11].

2. Informal description of G_0

In this short section we give an outline of the construction of G_0. We shall refer to Szemerédi's Regularity Lemma and its algorithmic version (Lemmas 2 and 3) and the packing theorem of Frankl and Rödl (see Lemma 5).

For an edge-weighted graph with weight function ω, we say that $\psi : \binom{G}{H_0} \to [0,1]$ is a fractional H_0-packing if $\sum_{H \ni e} \psi(H) \leq \omega(e)$ for each edge e. The basic approach to constructing a finite weighted graph G' from G for which $\nu^*_{H_0}(G')(|V(G)|/|V(G')|)^2$ is an approximate lower bound for $\nu_{H_0}(G)$ will be as follows. For some small quantity ϵ, we apply Szemerédi's lemma to G to get an ϵ-regular partition $V_0,\dots,V_s$, where $s \le S(\epsilon)$. We then define a finite graph G' with vertex set $\{1,\dots,s\}$, where ij is an edge of G' with weight d_{ij} precisely when (V_i,V_j) is an ϵ-regular pair of density $d_{ij} > 0$.

Let ψ^* be a maximum fractional H_0-packing of G'. Our aim is to convert ψ^* into an H_0-packing of G of size close to $|\psi^*|(|V(G)|/s)^2 = \nu^*_{H_0}(G')(|V(G)|/s)^2$. For each edge $e = ij$ of G', the edge set E_{ij} of the corresponding ϵ-regular pair (V_i,V_j) in G can be "sliced" into 2ϵ-regular graphs $E_{ij}(H)$, one for each copy H of H_0 in G' that contains e, such that $E_{ij}(H)$ has density $\psi^*(H)$ (see Lemma 6). Therefore each copy H of H_0 in G' with vertex set $\{i_1,\dots,i_h\}$, $h = |V(H_0)|$, corresponds to a subgraph of G with vertex set $\bigcup_{t=1}^h V_{i_t}$ and edge set $\bigcup_{ij\in E(H)} E_{ij}(H)$ where $E_{ij}(H)$ is 2ϵ-regular of density $\psi^*(H)$.

In such a subgraph, each edge is in approximately the same number of copies of H_0 in G. This is precisely the situation in which the packing theorem of Frankl and Rödl (see Lemma 5) applies, so each such subgraph contains a nearly-perfect packing of copies of H_0 in G. The union of all these packings is then an H_0-packing in G of the required size.

The problem with this approach is that the values $\{\psi^*(H)\}_{H\ni e}$ could be very small, say of order $|V(G')|^{-1}$ which can be much smaller than ϵ. Therefore, after slicing the density of each pair would be so small that no meaningful regularity would be preserved.

To overcome this difficulty we first prove that there exists a constant τ, depending only on H_0 and the error η, such that for any large enough weighted graph G', there exists a fractional H_0-packing ψ of G' with the following properties. The value $|\psi|$ is very close to $\nu^*_{H_0}(G')$, and for each $H \in \binom{G'}{H_0}$, if $\psi(H) > 0$ then $\psi(H) \ge \tau$. Then we may use the above approach, provided we choose the parameter ϵ to be much smaller than τ. See Lemma 7 for details.

3. Construction of G_0

Here we give the formal description of the finite weighted graph G_0. First we describe the Regularity Lemma of Szemerédi. Let a graph G with n vertices be fixed. For $U, W \subset V = V(G)$ with $U \cap W = \emptyset$, we write $E(U,W) = E_G(U,W)$ for the set of edges of G that have one endvertex in U and the other in W, and $G[U,W]$ for the subgraph of G with vertex set $U \cup W$ and edge set $E_G(U,W)$. We let the *density* $d(U,W)$ of the pair (U,W) be defined by $d(U,W) = |E(U,W)|/|U||W|$. Suppose $\epsilon > 0$. We say that the pair (U,W) is *ϵ-regular for* G if for all $U' \subset U$, $W' \subset W$ with $|U'| \ge \epsilon|U|$ and $|W'| \ge \epsilon|W|$, we have $|d(U',W') - d(U,W)| < \epsilon$.

We say that a partition $P=(V_i)_0^s$ of $V=V(G)$ is (ϵ,s)-*equitable* if $|V_0|\leq \epsilon n$, and $|V_1|=\ldots=|V_s|$. Then Szemerédi's Lemma [13] is as follows.

Lemma 2. *Let a real number $\epsilon>0$ and a positive integer s_0 be given. Then there exists a constant $S_0=S_0(\epsilon,s_0)\geq s_0$ such that for any graph G there exists an (ϵ,s)-equitable partition $P=(V_i)_0^s$ of $V(G)$, where $s_0\leq s\leq S_0$, such that all but at most $\epsilon\binom{s}{2}$ pairs (V_i,V_j) with $1\leq i<j\leq s$ are ϵ-regular.*

We shall use the following result. The statement is a straightforward generalization of Lemma 2, and the algorithmic part follows in a natural way from the algorithmic version of Lemma 2 proved by Alon, Duke, Lefmann, Rödl and Yuster [1].

Lemma 3. *Given a real number $\epsilon>0$ and positive integers q and s_0, there exists a constant $S_0=S_0(\epsilon,q,s_0)\geq s_0$ for which the following holds. Let G be a graph whose vertex set is partitioned into q classes $W_1,\ldots,W_q$. Then there exists an (ϵ,s)-equitable partition $P=(V_i)_0^s$ of $V(G)$, where $s_0\leq s\leq S_0$, such that each V_i for $i\geq 1$ is entirely contained in W_j for some $j=j(i)$, and all but at most $\epsilon\binom{s}{2}$ pairs (V_i,V_j) with $1\leq i<j\leq s$ are ϵ-regular. Moreover the partition P can be found in $O(M(n))$ time, where $n=|V(G)|$ and $M(n)$ is the time required to multiply two $n\times n$ matrices with 0,1 entries over the integers.*

Our next aim is to define the constants needed for the application of the Regularity Lemma we shall require. In order to do this we shall need the following four lemmas.

For a graph G and a partition $Q=(V_i)_{i=1}^k$ of $V(G)$, we say that a subgraph H of G is *crossing* in Q if no two vertices of H are in the same class V_i of Q. We write $\mathcal{C}(G,Q,H_0)$ for the set of copies of H_0 in G which are crossing in Q.

Lemma 4. *Let $\lambda>0$ and a graph H_0 be given. Then there exists $K=K(\lambda,H_0)$ such that the following holds.*

Let G be a graph with n vertices, and let ψ^* be a fractional H_0-packing of G. Then there exists a partition $Q=(V_i)_{i=1}^q$ of $V(G)$, where $q\leq K$, such that

(i) $\lfloor n/q\rfloor\leq|V_i|\leq\lceil n/q\rceil$ *for* $1\leq i\leq q$,
(ii) $\sum_{H\in\mathcal{C}(G,Q,H_0)}\psi^*(H)\geq(1-\lambda)\sum_{H\in\binom{G}{H_0}}\psi^*(H)$.

Moreover there is a $O(n^2)$ algorithm which finds such a partition.

We say that a pair (A,B) of disjoint vertex subsets of a graph G is ϵ-*regular with density* $\alpha\pm\epsilon$ if $|d_G(A',B')-\alpha|<\epsilon$ for every $A'\subset A$ and $B'\subset B$ with $|A'|\geq\epsilon|A|$ and $|B'|\geq\epsilon|B|$.

Let H be a graph with vertices $w_1,\dots,w_h$ that are ordered by $w_1<\dots<w_h$. Let G be an h-partite graph with vertex classes $V_1,\dots,V_h$. Then we say that a subgraph J of G with ordered vertex set $v_1<\dots<v_h$ is *partite-isomorphic* to H *in* $G[V_1,\dots,V_h]$ if $v_i\in V_i$ for each i and the map $v_i\to w_i$, $1\le i\le h$, is an isomorphism from J to H.

Lemma 5. *Let H be a graph with vertices $w_1,\dots,w_h$. Let real numbers $\lambda>0$ and $\phi>0$ be given. Then there exist $\theta=\theta(\lambda,\phi)$ and $B=B(\lambda,\phi)$ such that the following holds. Let G be an h-partite graph with vertex classes $V_1,\dots,V_h$ satisfying*

(i) *$|V_i|=k\ge B$ for each i,*
(ii) *for each $w_iw_j\in E(H)$ we have that (V_i,V_j) is θ-regular with density $\phi\pm\theta$,*
(iii) *for each $w_iw_j\notin E(H)$ we have that $V_i\cup V_j$ is an independent set.*

Then G contains a family of edge-disjoint subgraphs of G, each of which is partite-isomorphic to H in $G[V_1,\dots,V_h]$, which covers all but at most $\lambda|E(G)|$ edges of G. Moreover such a family can be found in time polynomial in $|V(G)|$.

The next lemma is concerned with partitioning a regular pair into edge-disjoint subgraphs of smaller density which are themselves regular.

Lemma 6. *Let $\xi'>0$ and $\sigma>0$ be given. Then there exists $\xi=\xi(\xi',\sigma)$ and $k_0=k_0(\xi',\sigma)$ such that the following holds. Let $j\le\sigma^{-1}$ be an integer and let G be a bipartite ξ-regular graph with bipartition (U,V), where $|U|=|V|=k\ge k_0$, and with density $d(U,V)$ where $j\sigma\le d(U,V)<(j+1)\sigma$. Then there is a $O(k^2)$ algorithm which finds edge-disjoint subgraphs $G^1,\dots,G^j$ in G such that each G^i is ξ'-regular of density $\sigma\pm\xi'$.*

For a weighted graph G and a real number $\tau>0$, we say a fractional H_0-packing ψ of G is *τ-bounded* if for each $H\in\binom{G}{H_0}$, either $\psi(H)=0$ or $\psi(H)\ge\tau$.

Lemma 7. *Let a graph H_0 and a real number $\eta>0$ be given. Then there exists $\tau=\tau(H_0,\eta)$ such that the following holds. For every weighted graph G with n vertices, there exists a τ-bounded fractional H_0-packing ψ of G such that $|\psi|\ge\nu^*_{H_0}(G)-\eta n^2$.*

The proof of Lemma 7 is not algorithmic, but we shall apply it only in a finite weighted graph. Therefore we are not concerned with its algorithmic aspects, as in this setting a suitable ψ can be found by exhaustive search. We mention that it is possible to give an algorithmic proof of this lemma, using the Regularity Lemma and Lemma 5. The proofs of Lemmas 4, 5, 6 and 7 appear in Sections 5, 6, 7 and 9 respectively.

The proof of Theorem 1 involves the construction of a finite weighted graph from the given G, and the use of a fractional H_0-packing in the finite graph to construct a packing in G. The properties of this weighted graph are made explicit in the next lemma, the proof of which appears in Section 8.

Lemma 8. *Let H_0 be a fixed graph, let an integer $s_0 \geq 0$ and real numbers α and ϵ' be given. Then there exists $S = S(H_0, \alpha, \epsilon', s_0)$ such that the following holds. Let G be a graph with n vertices. Then G has a subgraph $\bar{G}$ with the following properties.*

(i) *$V(\bar{G})$ has a partition $P = C_1 \cup \ldots \cup C_s$ where each C_i is an independent set, $s_0 \leq s \leq S$, and $m = |C_1| = \ldots = |C_s| \geq (1-\epsilon')n/s$.*
(ii) *Each pair (C_i, C_l) induces an ϵ'-regular bipartite subgraph $J(i,l)$ of $\bar{G}$, with some density d_{il}.*
(iii) *$|E(G)| - |E(\bar{G})| \leq 3\epsilon' n^2$.*
Let G_0 be the finite weighted graph with vertex set $\{c_1, \ldots, c_s\}$, where each pair c_i, c_l is joined by an edge of weight d_{il}. Then G_0 satisfies
(iv) *$m^2 \nu^*_{H_0}(G_0) \geq \nu^*_{H_0}(G) - \alpha n^2$.*

Moreover $\bar{G}$ and G_0 can be found in polynomial time.

We may now describe the construction of G_0. Let H_0 and η be fixed, and let $h = |V(H_0)|$, $r = |E(H_0)|$. Clearly we may assume $\eta < 1/2$. We now define a collection of constants, some of which will be referred to in the proof of Theorem 9 below.

We let $\alpha = \eta/2$, (see Lemma 8),
$\tau = \tau(H_0, \eta/10)$ (see Lemma 7),
$\beta = \tau\eta/5$,
$\epsilon'' = \min\{\theta(\eta/10, \beta), \beta\eta/5\}$ (see Lemma 5),
$\epsilon' = \xi(\epsilon'', \beta)$ (see Lemma 6), and
$N = \max\{(1-\epsilon')^{-1} S(H_0, \alpha, \epsilon', 1) k_0(\epsilon'', \beta), (1-\epsilon')^{-1} S(H_0, \alpha, \epsilon', 1) B(\eta/10, \beta)\}$ (see Lemmas 5, 6 and 8).

These constants satisfy

$$\eta > \alpha \gg \tau \gg \beta \gg \epsilon'' \gg \epsilon' \gg 1/N.$$

Let the graph G with n vertices be given, where $n \geq N$. Then we apply Lemma 8 to G with parameters H_0, α, ϵ', and 1 to obtain the graph $\bar{G}$ and the finite weighted graph G_0. Then we have the following.

Theorem 9. *With the above definitions we have*

(i) $\nu^*_{H_0}(G) \leq \nu^*_{H_0}(G_0)m^2 + \eta n^2/2$,
(ii) $\nu_{H_0}(G) \geq \nu^*_{H_0}(G_0)m^2 - \eta n^2/2$.

Moreover an H_0-packing in G of size at least $\nu^*_{H_0}(G_0)m^2 - \eta n^2/2$ can be found in time polynomial in n.

Proof of Theorem 1. The algorithm from Theorem 9 finds an H_0-packing in G of size at least $\nu^*_{H_0}(G_0)m^2 - \eta n^2/2$ which by Theorem 9(i) is at least $\nu^*_{H_0}(G) - \eta n^2/2 - \eta n^2/2 \geq \nu^*_{H_0}(G) - \eta n^2$. Therefore the theorem is proved. ∎

The complexity of the algorithm from Theorem 9 depends on the complexity of Lemma 5, which in turn depends on that of Grable's algorithm (see [11]).

4. Proof of Theorem 9

Part (i) of Theorem 9 is immediate from (iv) of Lemma 8. Therefore we consider the proof of Part (ii). Recall that G_0 is the weighted graph with vertex set $\{c_1, \dots, c_s\}$ with the properties listed in Lemma 8. Then since $\tau = \tau(H_0, \eta/10)$, by Lemma 7 there exists a τ-bounded fractional H_0-packing ψ of G_0 satisfying

$$\nu^*_{H_0}(G_0) - \eta s^2/10 \leq |\psi| = \sum_{H \in \binom{G_0}{H_0}} \psi(H). \tag{1}$$

Then $\sum_{H \ni e} \psi(H) \leq 1$ for every $e \in E(G_0)$, and for each $H \in \binom{G_0}{H_0}$ we have $\psi(H) = 0$ or $\psi(H) \geq \tau$. Our aim is to find an H_0-packing in G of size approximately $\nu^*_{H_0}(G_0)m^2$. The proof will be algorithmic, and at the end of this section we shall explicitly state the algorithm for finding this packing, which will also include the construction from Lemma 8.

First we modify ψ slightly, as we would like the value it takes on each $H \in \binom{G_0}{H_0}$ to be an integer multiple of β. Therefore we define $\bar{\psi} : \binom{G_0}{H_0} \to [0,1]$ by $\bar{\psi}(H) = i_H \beta$ where $i_H = \left\lfloor \frac{\psi(H)}{\beta} \right\rfloor$. Note then that

$$\begin{aligned} |\psi| - |\bar{\psi}| &\leq \sum_{H \in \binom{G_0}{H_0}} (\psi(H) - \bar{\psi}(H)) \\ &\leq \beta \left|\left\{H \in \binom{G_0}{H_0} : \psi(H) > 0\right\}\right| \leq \beta \tau^{-1} |\psi|. \end{aligned} \tag{2}$$

The rest of the proof will consist of the following three steps. In Step 1, for each $e=c_ic_l\in E(G_0)$, we shall slice $J(i,l)$ (see Lemma 8 (ii)) into edge-disjoint regular subgraphs of density about β which we shall call *elementary* subgraphs of G. The aim of Step 2 is to assign to each edge of each $H\in\binom{G_0}{H_0}$ the appropriate number i_H of elementary subgraphs of the corresponding regular pair in G. Then in Step 3 we show that for each H the union of its assigned elementary subgraphs in G contains a nearly-perfect packing of copies of H_0 in G.

Now we describe Step 1. Let $e=c_ic_l\in E(G_0)$ be an edge of G_0. Consider the subgraph $J(i,l)$ of G from part (ii) of Lemma 8. Then $J(i,l)$ is ϵ'-regular of density d_{il}, with vertex classes $C_i\cup C_l$. We wish to apply Lemma 6 to $J(i,l)$ with parameters ϵ'' (for ξ') and β (for σ). Note that by definition of N and Lemma 8 we have $|C_i|=|C_l|=m\geq(1-\epsilon')N/S\geq k_0(\epsilon'',\beta)$, where $S=S(H_0,\alpha,\epsilon',1)$ is as in Lemma 8. We also have (see definition of ϵ') that $\epsilon'\leq\xi(\epsilon'',\beta)$. Therefore we may apply Lemma 6 to $J(i,l)$ to find x^e edge-disjoint subgraphs $I_e^1,\ldots,I_e^{x^e}$ such that each is ϵ''-regular of density $\beta\pm\epsilon''$, where $x^e\geq\lfloor d_{il}/\beta\rfloor$. These are the subgraphs we call elementary subgraphs of G. We repeat this for each edge $e\in E(G_0)$. This completes Step 1.

Next we turn to Step 2. Let an edge $e=c_ic_l$ of G_0 be fixed. We assign to each $H\in\binom{G_0}{H_0}$ such that $e\in H$ a set $S_e(H)$ of i_H elementary subgraphs of $J(i,l)$ such that the $\{S_e(H)\}_{H\ni e}$ are all edge-disjoint. This is possible since $\sum_{H\ni e}i_H\leq\sum_{H\ni e}\lfloor\psi(H)/\beta\rfloor\leq\lfloor d_{il}/\beta\rfloor\leq x^e$. We repeat this for every edge of G_0. Then each $H\in\binom{G_0}{H_0}$ with vertex set $\{c_{i_1},\ldots,c_{i_h}\}$ is assigned i_H elementary subgraphs contained in $J(i_v,i_p)$ for each edge $c_{i_v}c_{i_p}\in E(H)$. This completes Step 2.

Finally we consider Step 3. Let $H\in\binom{G_0}{H_0}$ be fixed, and let its vertex set be $\{c_{i_1},\ldots,c_{i_h}\}$. Let L be a subgraph of G with vertex set $C_{i_1}\cup\ldots\cup C_{i_h}$ formed by placing one elementary graph assigned to H on each pair (C_{i_v},C_{i_p}) for which $c_{i_v}c_{i_p}$ is an edge of H. Then since each vertex class of L has size $m\geq(1-\epsilon')N/S\geq B(\eta/10,\beta)$ (see the definition of N), and $\epsilon''\leq\theta(\eta/10,\beta)$, we may apply Lemma 5 to L with parameters $\eta/10$ and β to find a family of edge-disjoint copies of H_0 in L that covers at least $(1-\eta/10)|E(L)|\geq(1-\eta/10)r(\beta-\epsilon'')m^2$ edges of L (recall that $|E(H_0)|=r$), and hence has size at least $(1-\eta/10)(\beta-\epsilon'')m^2$.

Since all the elementary subgraphs were edge-disjoint, we can construct i_H edge-disjoint graphs L of this type associated with H. Therefore we find altogether an H_0-packing in G corresponding to H of size at least $(1-\eta/10)i_H(\beta-\epsilon'')m^2$. Now summing over all $H\in\binom{G_0}{H_0}$ we find an H_0-packing in G of size at least

$$\begin{aligned}
&(1-\eta/10)\sum_{H\in\binom{G_0}{H_0}} i_H(\beta-\epsilon'')m^2 \\
&\geq (1-\eta/10-\epsilon''/\beta)|\bar{\psi}|m^2 \\
&\geq (1-\eta/10-\epsilon''/\beta)(1-\beta/\tau)|\psi|m^2 && \text{by (2)} \\
&\geq |\psi|m^2-(\eta/10+\epsilon''/\beta+\beta/\tau)|\psi|m^2 \\
&\geq \nu^*_{H_0}(G_0)m^2-\eta s^2m^2/10-(\eta/10+\epsilon''/\beta+\beta/\tau)\nu^*_{H_0}(G_0)m^2 && \text{by (1)} \\
&\geq \nu^*_{H_0}(G_0)m^2-\eta n^2/10-(\eta/10+\epsilon''/\beta+\beta/\tau)n^2/2,
\end{aligned}$$

where the last line follows since $\nu^*_{H_0}(G_0)\leq\binom{s}{2}<s^2/2$ and $s^2m^2\leq n^2$. Then since $\epsilon''/2\beta\leq\eta/10$ and $\beta/2\tau\leq\eta/10$ (see the definitions of these constants), we conclude that $\nu_{H_0}(G)\geq\nu^*_{H_0}(G_0)m^2-\eta n^2/2$ as required. ∎

To end this section we summarise the whole argument in the following algorithm for finding an H_0-packing in G of nearly optimal size.

Algorithm
Input: Graph G with $|V(G)|=n\geq N$, real $\eta>0$, and maximum fractional H_0-packing ψ^* of G.
Construct G_0 from G using Lemma 8 (see Section 9 for definitions):
Apply Lemma 4 to G and ψ^* to obtain the partition $Q=(V_i)_{i=1}^q$ of $V(G)$.
Apply Lemma 3 to G and Q to obtain a partition $(C_u)_{u=0}^s$ of $V(G)$.
Form the weighted graph G_0.
Find a τ-bounded fractional packing of G_0:
For $\tau=\tau(H_0,\eta/10)$, let ψ be a τ-bounded fractional H_0-packing of G_0 satisfying $|\psi|\geq\nu^*_{H_0}(G_0)-\eta s^2/10$ (as guaranteed by Lemma 7). Form $\bar{\psi}$.
Form elementary subgraphs in G:
For each $e=c_ic_l\in E(G_0)$,
Apply Lemma 6 to $J(i,l)$ to slice it into ϵ''-regular elementary subgraphs $I_e^1,\dots,I_e^{x^e}$ of density $\beta\pm\epsilon''$.
For each $H\in\binom{G_0}{H_0}$, assign i_H elementary subgraphs to each of its edges:
For each edge $e=c_ic_l\in E(G_0)$,
assign a set $S_e(H)$ of i_H elementary subgraphs of $J(i,l)$ to e.
For each $H\in\binom{G_0}{H_0}$, find a nearly-perfect packing of copies of H_0 in the union of the elementary subgraphs assigned to H:
For each H, let $\{c_{t_1},\dots,c_{t_h}\}$ be the vertex set of H.
Form a subgraph L by placing an elementary graph assigned to H on each (C_{i_v},C_{i_p}) for which $c_{i_v}c_{i_p}\in E(H)$.
Apply Lemma 5 to L to find a nearly-perfect packing of copies of H_0 in L.

Repeat this for a total of i_H edge-disjoint subgraphs L.
The union of all these packings over all $H \in \binom{G_0}{H_0}$ gives an H_0-packing in G of value at least $\nu^*_{H_0}(G_0)m^2 - \eta n^2/2$.

5. Proof of Lemma 4

First we prove the following easy result for partitioning weighted sets.

Lemma 10. *Let V be a set and $w : \binom{V}{2} \to \mathbb{R}^+$ be a non-negative weight function defined on the pairs of elements of V. Let $|V| = n$. Then there is a $O(n^2)$ algorithm which partitions V into subsets V_0 and V_1 such that $\lfloor n/2 \rfloor \le |V_0|, |V_1| \le \lceil n/2 \rceil$ and*

$$\sum_{x,y \in V_0} w(x,y) + \sum_{x,y \in V_1} w(x,y) \le \frac{1}{2} \sum_{x,y \in V} w(x,y).$$

Proof. We construct the partition greedily. First we take two arbitrary elements a and b of V and let $A_1 = \{a\}$ and $B_1 = \{b\}$. Now suppose that $i \ge 1$ and sets A_i and B_i have been defined, where $|A_i| = |B_i| = i$ and

$$\sum_{x,y \in A_i} w(x,y) + \sum_{x,y \in B_i} w(x,y) \le \frac{1}{2} \sum_{x,y \in A_i \cup B_i} w(x,y). \tag{3}$$

Suppose also that $|V \setminus (A_i \cup B_i)| \ge 2$. Then we define A_{i+1} and B_{i+1} by taking two arbitrary elements a and b of $V \setminus (A_i \cup B_i)$ and letting $A_{i+1} = A_i \cup \{a\}$ and $B_{i+1} = B_i \cup \{b\}$ if

$$\sum_{v \in B_i} w(a,v) + \sum_{v \in A_i} w(b,v) \ge \sum_{v \in A_i} w(a,v) + \sum_{v \in B_i} w(b,v),$$

otherwise we let $A_{i+1} = A_i \cup \{b\}$ and $B_{i+1} = B_i \cup \{a\}$. Then note that (3) still holds for the sets A_{i+1} and B_{i+1}.

We continue this procedure until $|V \setminus (A_i \cup B_i)| \le 1$. If n is odd then we add the last vertex a to A_i if $\sum_{v \in B_i} w(a,v) \ge \sum_{v \in A_i} w(a,v)$, otherwise we add it to B_i. Then (3) still holds, so letting V_0 and V_1 denote the final sets obtained in this construction completes the proof. ∎

The proof of Lemma 4 will follow immediately from the following more general result. Here for a partition $Q = (V_i)_{i=1}^q$ of a set V, we say that a subset S of V is *crossing* in Q if $|S \cap V_i| \le 1$ for each i, and we write $\mathcal{C}(t, V, Q)$ for the set of subsets of V of size t that are crossing in Q.

Lemma 11. *Let a positive integer t and a real number $\lambda > 0$ be given. Then there exists $K = K(t,\lambda)$ such that the following holds. For any set V and function $f : \binom{V}{t} \to \mathbb{R}^+$, there exists a partition $Q = (V_i)_{i=1}^q$ with $q \le K$ such that*

(i) $||V_i| - |V_j|| \le 1$ *for each* i, j,
(ii) $\sum_{S \in \mathcal{C}(t,V,Q)} f(S) \ge (1-\lambda) \sum_{S \in \binom{V}{t}} f(S)$.

Proof. Let $n = |V|$, let k be such that $2^k > \binom{t}{2}\lambda^{-1}$ and let $K = 2^k$. If $n \le K$ then the partition of V into singletons clearly satisfies the required conditions, so we may assume $n > K$. We define a weight function w on $\binom{V}{2}$ by $w(x,y) = \sum_{S \in \binom{V}{t} : S \ni x,y} f(S)$, and we apply Lemma 10 to V to obtain a partition $A_0 \cup A_1$ such that $\lfloor n/2 \rfloor \le |A_0|, |A_1| \le \lceil n/2 \rceil$ and

$$\sum_{x,y \in A_0} w(x,y) + \sum_{x,y \in A_1} w(x,y) \le \frac{1}{2} \sum_{x,y \in V} w(x,y).$$

Now suppose $i \ge 1$ and that we have found a partition $V = \bigcup_{\sigma \in \{0,1\}^i} A_\sigma$ into subsets such that for each σ, $\lfloor n/2^i \rfloor \le |A_\sigma| \le \lceil n/2^i \rceil$ and

$$\sum_{\sigma \in \{0,1\}^i} \sum_{x,y \in A_\sigma} w(x,y) \le \frac{1}{2^i} \sum_{x,y \in V} w(x,y).$$

We apply Lemma 10 again to each A_σ to obtain a partition $A_\sigma = A_{\sigma 0} \cup A_{\sigma 1}$ such that $\lfloor n/2^{i+1} \rfloor \le |A_{\sigma 0}|, |A_{\sigma 1}| \le \lceil n/2^{i+1} \rceil$ and

$$\sum_{x,y \in A_{\sigma 0}} w(x,y) + \sum_{x,y \in A_{\sigma 1}} w(x,y) \le \frac{1}{2} \sum_{x,y \in A_\sigma} w(x,y).$$

Therefore after k steps we get a partition $V = \bigcup_{\sigma \in \{0,1\}^k} A_\sigma$ such that $\lfloor n/2^k \rfloor \le |A_\sigma| \le \lceil n/2^k \rceil$ and

$$\sum_{\sigma \in \{0,1\}^k} \sum_{x,y \in A_\sigma} w(x,y) \le \frac{1}{2^k} \sum_{x,y \in V} w(x,y) = \frac{1}{2^k} \binom{t}{2} \sum_{S \in \binom{V}{t}} f(S).$$

We let Q be this partition. Then note that

$$\sum_{S \notin \mathcal{C}(t,V,Q)} f(S) \le \sum_{\sigma \in \{0,1\}^k} \sum_{x,y \in A_\sigma} w(x,y) \le \lambda \sum_{S \in \binom{V}{t}} f(S).$$

This completes the proof. ∎

Proof of Lemma 4. This follows immediately by taking $V = V(G)$, $t = h = |V(H_0)|$ and $f(S) = \sum_{H \in \binom{V(G)}{H_0} : V(H) = S} \psi^*(H)$. ∎

6. Proof of Lemma 5

In order to prove Lemma 5, we shall make use of the following three results. The first is the packing theorem of Frankl and Rödl [7]. By a *packing* of a hypergraph $\mathcal{H}$ we mean a set of pairwise disjoint edges of $\mathcal{H}$.

Theorem 12. *Let $a > 3$ be a real number. Let an integer r and a real number $\lambda > 0$ be given. Then there exists $\beta = \beta(r, \lambda) > 0$ such that the following holds. Let $\mathcal{H}$ be an r-uniform hypergraph with vertex set X, $|X| = n$, such that for some D*

(i) $(1-\beta)D < deg_{\mathcal{H}}(x) < (1+\beta)D$ *for all vertices* $x \in X$,
(ii) $deg_{\mathcal{H}}(x,y) < D/(\log n)^a$ *for all distinct* x, $y \in X$.

Then $\mathcal{H}$ has a packing which covers all but λn vertices in X.

There are many generalizations of this theorem by various authors, but this one is enough for our purposes.

The following algorithmic version of the above result was proved by Grable [11].

Theorem 13. *Given r and λ as above, the packing of $\mathcal{H}$ can be found in time polynomial in n.*

The next result is a lemma from [6].

Lemma 14. *Let H_0 be a fixed graph with vertex set $w_1, \ldots, w_h$, and let $\xi > 0$ be given. Let G be an h-partite graph with vertex classes $V_1, \ldots, V_h$ such that for each $i \neq j$ the pair (V_i, V_j) is ξ-regular with density d_{ij}. Let*

$$\hat{d}_{ij} = \begin{cases} d_{ij} & \text{if } w_i w_j \in E(H_0) \\ 1 - d_{ij} & \text{if } w_i w_j \notin E(H_0). \end{cases}$$

Then the number $f(H_0, G)$ of induced subgraphs of G which are partite-isomorphic to H_0 in $G[V_1, \ldots, V_h]$ satisfies

$$\Big| f(H_0, G) - \prod_{1 \le i < j \le h} \hat{d}_{ij} \prod_{1 \le i \le h} |V_i| \Big| \le (\xi h^3)^{1/2} \prod_{1 \le i \le h} |V_i|.$$

Our approach to proving Lemma 5 will be to apply Theorem 12 to the hypergraph whose vertex set is $E(G)$ and whose edges are the edge sets of subgraphs in G which are partite-isomorphic to H. Therefore we first need to estimate the number of such subgraphs which contain a given edge of G. This step is accomplished by the following lemma, whose proof depends on Lemma 14.

Lemma 15. *Let H be a graph with vertices $w_1,\dots,w_h$ and with r edges. Let real numbers $\lambda>0$ and $\phi>0$ be given. Then there exists $\zeta=\zeta(H,\lambda,\phi)$ such that the following holds. Let G be an h-partite graph with vertex classes $V_1,\dots,V_h$ satisfying*

(i) $|V_i|=b$ for each i,
(ii) for each $w_iw_j\in E(H)$ we have that (V_i,V_j) is ζ-regular with density $\phi\pm\zeta$,
(iii) for each $w_iw_j\notin E(H)$ we have that $V_i\cup V_j$ is an independent set.

Then there exists a subgraph $G''\subset G$ with vertex classes $V_1',\dots,V_h'$, $V_i'\subset V_i$, and at least $(1-\lambda)|E(G)|$ edges such that for every edge e of G'',

$$|q_H(e)-\phi^{r-1}b^{h-2}|<\lambda\phi^{r-1}b^{h-2},$$

where $q_H(e)$ denotes the number of subgraphs J of G'' containing e which are partite-isomorphic to H in $G''[V_1',\dots,V_h']$. Moreover there is a $O(M(b))$ algorithm to construct G''.

Proof. Let $\zeta>0$ be chosen to satisfy the following conditions:

(a) $(1-2h\zeta^{1/2}\phi^{-1})^r(1-2h\zeta)^h(1-4h\zeta\phi^{-1})^h(1-8h\zeta\phi^{-2})^h>1-\lambda/4$,
(b) $(1+2h\zeta^{1/2}\phi^{-1})^r(1+4h\zeta\phi^{-1})^h(1+8h\zeta\phi^{-2})^h<1+\lambda/6$,
(c) $\phi^{1-r}\left(\frac{\zeta^{1/2}h^3}{\phi^2-8h\zeta}\right)^{1/2}<\lambda/2$,
(d) $(\phi-4h\zeta)(1-2h\zeta)^2>(1-\lambda)(\phi+\zeta)$.

First note that for each i, j with $w_iw_j\in E(H)$, at most ζb vertices x of V_i satisfy $||\Gamma(x)\cap V_j|-\phi b|\ge\zeta b$. Therefore by removing all such vertices, we may construct, for each $i=1,\dots,h$, subsets $V_i'\subset V_i$ with $|V_i'|\ge(1-2h\zeta)b$ such that for every $x\in V_i'$ and every j such that $w_iw_j\in E(H)$ we have $||\Gamma(x)\cap V_j'|-\phi b|<2h\zeta b$. We now focus on G', the subgraph of G induced by $V_1'\cup\ldots\cup V_h'$.

Now suppose $\{w_i,w_j,w_l\}$ are vertices of a triangle T in H. We say that an edge xy of G' is *bad* for T if $x\in V_i'$ and $y\in V_j'$, and $|\Gamma(x)\cap\Gamma(y)\cap V_l'|>(\phi+2h\zeta)(\phi+2\zeta)b$ or $|\Gamma(x)\cap\Gamma(y)\cap V_l'|<(\phi-2h\zeta)(\phi-2\zeta)b$. For $x\in V_i'$, the set U of vertices $y\in\Gamma(x)\cap V_j'$ such that xy is bad for T is at most $2\zeta b$ by ζ-regularity. Therefore the subgraph of G' formed by the edges which are bad for some triangle T of H has maximum degree at most $2(h-2)(h-1)\zeta b$. Let G'' denote the subgraph of G' formed by removing all bad edges (the bad edges can all be identified by performing h matrix multiplications). Then G'' has the following properties.

(1) For each $x\in V_i'$ and each j with $w_iw_j\in E(H)$ we have $||\Gamma(x)\cap V_j'|-\phi b|<4h\zeta b$,

(2) for every triangle $\{w_i,w_j,w_l\}$ of H and each edge xy with $x\in V_i'$, $y\in V_j'$ we have $||\Gamma(x)\cap\Gamma(y)\cap V_l'|-\phi^2 b|<8h\zeta b$,
(3) each pair (V_i',V_j') where $w_iw_j\in E(H)$ is $\zeta^{1/2}$-regular with density $\phi\pm 2h\zeta^{1/2}$.
(4) $|E(G'')|\geq(1-\lambda)|E(G)|$ by (d).

We shall use Lemma 14 to show that these very regular properties of G'' imply the result. Let $e=xy$ be an edge of G'', and suppose $x\in V_1'$, $y\in V_2'$. Let us classify the indices i for $3\leq i\leq h$ into four sets as follows: for each subset $S\subseteq\{1,2\}$ we let $\mathcal{C}_S$ denote the set of indices i for which $w_\sigma w_i\in E(H)$ for all $\sigma\in S$ and $w_\sigma w_i\notin E(H)$ for all $\sigma\in\{1,2\}\setminus S$. Then we make the following definitions.

$$V_i(x,y)=\begin{cases} V_i' & \text{if } i\in\mathcal{C}_\emptyset\\ V_i'\cap\Gamma(x) & \text{if } i\in\mathcal{C}_{\{1\}}\\ V_i'\cap\Gamma(y) & \text{if } i\in\mathcal{C}_{\{2\}}\\ V_i'\cap\Gamma(x)\cap\Gamma(y) & \text{if } i\in\mathcal{C}_{\{1,2\}}.\end{cases}$$

Then the number of subgraphs of G'' partite-isomorphic to H in $G''[V_1',\dots,V_h']$ that contain e is precisely the number of copies of $H_{12}=H-\{w_1,w_2\}$ with vertices $\{v_3\dots,v_h\}$ with $v_i\in V_i(x,y)$ for each i, where the order of vertices in H_{12} is the same as in H. Note that by Properties (2) and (3), whenever $w_iw_j\in E(H)$, the pair $(V_i(x,y),V_j(x,y))$ is $\zeta^{1/2}(\phi^2-8h\zeta)^{-1}$-regular of density d_{ij} where $\phi-2h\zeta^{1/2}\leq d_{ij}\leq\phi+2h\zeta^{1/2}$.

Therefore by Lemma 14 and (c) we find that e satisfies

$$\Big|q_H(e)-\prod_{w_iw_j\in E(H_{12})}d_{ij}\prod_{3\leq i\leq h}|V_i(x,y)|\Big|\leq$$
$$\left(\frac{\zeta^{1/2}h^3}{\phi^2-8h\zeta}\right)^{1/2}\prod_{3\leq i\leq h}|V_i(x,y)|<\lambda\phi^{r-1}b^{h-2}/2.$$

Moreover we have

$$(\phi-2h\zeta^{1/2})^{|E(H_{12})|}\leq\prod_{w_iw_j\in E(H_{12})}d_{ij}\leq(\phi+2h\zeta^{1/2})^{|E(H_{12})|},$$

and from Properties (1) and (2) and the fact that $|V_i'|\geq(1-2h\zeta)b$ we see

$$\prod_{3\leq i\leq h}|V_i(x,y)|\geq b^{h-2}(1-2h\zeta)^{|\mathcal{C}_\emptyset|}(\phi-4h\zeta)^{|\mathcal{C}_{\{1\}}\cup\mathcal{C}_{\{2\}}|}(\phi^2-8h\zeta)^{|\mathcal{C}_{\{1,2\}}|}$$
$$\geq\phi^{|\mathcal{C}_{\{1\}}|+|\mathcal{C}_{\{2\}}|+2|\mathcal{C}_{\{1,2\}}|}b^{h-2}(1-2h\zeta)^{|\mathcal{C}_\emptyset|}(1-4h\zeta\phi^{-1})^{|\mathcal{C}_{\{1\}}\cup\mathcal{C}_{\{2\}}|}(1-8h\zeta\phi^{-2})^{|\mathcal{C}_{\{1,2\}}|}$$

and

$$\prod_{3\le i\le h} |V_i(x,y)| \le b^{h-2}(\phi+4h\zeta)^{|\mathcal{C}_{\{1\}}\cup\mathcal{C}_{\{2\}}|}(\phi^2+8h\zeta)^{|\mathcal{C}_{\{1,2\}}|}$$
$$\le \phi^{|\mathcal{C}_{\{1\}}|+|\mathcal{C}_{\{2\}}|+2|\mathcal{C}_{\{1,2\}}|}b^{h-2}(1+4h\zeta\phi^{-1})^{|\mathcal{C}_{\{1\}}\cup\mathcal{C}_{\{2\}}|}(1+8h\zeta\phi^{-2})^{|\mathcal{C}_{\{1,2\}}|}.$$

Therefore by (a) and (b) and the fact that $|E(H_{12})|+|\mathcal{C}_{\{1\}}|+|\mathcal{C}_{\{2\}}|+2|\mathcal{C}_{\{1,2\}}|=r-1$ we find

$$\Big|\prod_{w_iw_j\in E(H_{12})} d_{ij} \prod_{3\le i\le h} |V_i(x,y)| - \phi^{r-1}b^{h-2}\Big| < \lambda\phi^{r-1}b^{h-2}/2$$

which implies the result. ∎

We are now ready to prove Lemma 5.

Proof of Lemma 5. Let r denote the number of edges of H. Given G, let G'' denote the graph given by the algorithm in Lemma 15, and let the r-uniform hypergraph $\mathcal{H}$ be defined as follows. The vertex set $V(\mathcal{H})=E(G'')$, and a set of r edges of G'' forms an edge of $\mathcal{H}$ if it is the edge set of a subgraph of G'' which is partite-isomorphic to H_0 in $G''[V'_1,\dots,V'_h]$.

Let $\theta=\zeta(H_0,\beta(r,\lambda/2),\phi)$. Then by Lemma 15 we have that $deg_{\mathcal{H}}(e)=q_{H_0}(e)$ satisfies

$$|deg_{\mathcal{H}}(e)-\phi^{r-1}k^{h-2}| < \beta(r,\lambda/2)\phi^{r-1}k^{h-2}$$

for every element e of $V(\mathcal{H})$. Moreover we note that for any two distinct e_1, $e_2\in V(\mathcal{H})$ we have $deg_{\mathcal{H}}(e_1,e_2)<k^{h-3}$. Therefore, provided B is large enough such that $(\log(hk))^4<\phi^{r-1}k$ for all $k\ge B$, by Theorem 12 we have that $\mathcal{H}$ contains a packing which covers at least $(1-\lambda/2)|V(\mathcal{H})|\ge(1-\lambda)|E(G)|$ elements of $V(\mathcal{H})$, *i.e.* the conclusion of the lemma holds. By Theorem 13 this packing can be found in time polynomial in $|V(G)|$. ∎

7. Proof of Lemma 6

It would be quite easy to give a non-algorithmic proof of Lemma 6, essentially just by partitioning the edges of G into j classes at random. Our plan for proving it algorithmically is to partition the vertex sets U and V into a large but finite number ℓ of classes, and then define $G^1,\dots,G^j$ using the following lemma for partitioning the complete bipartite graph $K_{\ell,\ell}$, treating the regular pairs joining vertex classes in G as edges of $K_{\ell,\ell}$.

Lemma 16. *For every integer $s \geq 1$ and real number $\rho > 0$ there exists an integer $\ell = \ell(s,\rho)$ such that the following holds. For every β, $0 \leq \beta < 1$, the complete bipartite graph with bipartition $A \cup B$ where $|A| = |B| = \ell$ contains edge-disjoint subgraphs $J_1, \dots, J_s$ with the following property. For every function $w : A \cup B \to [0,1]$ with $w(A) = \sum_{a\in A} w(a) \geq \rho\ell$ and $w(B) = \sum_{b\in B} w(b) \geq \rho\ell$ and every i with $1 \leq i \leq s$ we have*

$$\left(\frac{1}{s+\beta} - \rho\right) w(A)w(B) \leq \sum_{ab\in E(J_i)} w(a)w(b) \leq \left(\frac{1}{s+\beta} + \rho\right) w(A)w(B).$$

Proof. Let J be any fixed bipartite graph with bipartition $A\cup B$, and let a function w satisfying the conditions be given. Our first aim is to show that there exists $w_0 : A \cup B \to [0,1]$ such that

(i) $w_0(A) = w(A)$ and $w_0(B) = w(B)$,
(ii) $W(w_0) = \sum_{ab\in E(J)} w_0(a)w_0(b) \geq W(w)$,
(iii) the sets $M_A(w_0) = \{a \in A : 0 < w(a) < 1\}$ and $M_B(w_0) = \{b \in B : 0 < w(b) < 1\}$ satisfy $w_0(M_A(w_0)) \leq 1$ and $w_0(M_B(w_0)) \leq 1$.

Suppose that w itself does not satisfy Property (iii), so without loss of generality suppose $w(M_A(w)) > 1$. Let $M_A(w) = \{a_0, \dots, a_r\}$ and assume that $w(\Gamma(a_0)) \geq \dots \geq w(\Gamma(a_r))$. Then there exist $\epsilon_1, \dots, \epsilon_r > 0$ such that $1 - w(a_0) = \sum_{j=1}^{r} \epsilon_j$ and $w(a_j) - \epsilon_j \geq 0$ for $1 \leq j \leq r$. Let $w' : A \cup B \to [0,1]$ be defined by $w'(c) = w(c)$ for $c \in (A \setminus M_A(w)) \cup B$, $w'(a_j) = w(a_j) - \epsilon_j$ for $j = 1, \dots, r$, and $w'(a_0) = 1$. Then note that

- $w'(A) = w(A)$ and $w'(B) = w(B)$,
- $W(w') - W(w) = \sum_{a\in M_A(w)} (w'(a) - w(a))w(\Gamma(a)) = (1 - w(a_0))w(\Gamma(a_0)) - \sum_{j=1}^{r} \epsilon_j w(\Gamma(a_j)) \geq 0$,
- $w'(M_A(w')) = w(M_A(w)) - 1$ and $w'(M_B(w')) = w(M_B(w))$.

Therefore repeating this argument shows that w_0 exists as claimed.

In order to construct the graphs $J_1, \dots, J_s$ we recall a few facts from random graph theory. Let $\mathbb{J}(A,B,s,\beta)$ denote a random partition of the set $A \times B$ into $s+1$ parts $\mathbb{J}_0, \dots, \mathbb{J}_s$, where for each pair (a,b) we let $\mathbb{P}[(a,b) \in \mathbb{J}_0] = \beta/(s+\beta)$ and $\mathbb{P}[(a,b) \in \mathbb{J}_i] = 1/(s+\beta)$, $1 \leq i \leq s$, independently of all other pairs. By Chernoff's inequality we find that for large enough $\ell = \ell(s,\rho)$ and every $0 \leq \beta < 1$, the following event occurs with positive probability. For every $A' \subseteq A$, $B' \subseteq B$ with $|A'| \geq \rho\ell/2$, $|B'| \geq \rho\ell/2$ and $i = 1 \dots, s$ we have

$$|\mathbb{J}_i \cap (A' \times B')| \leq \frac{1}{s+\beta}\left(1 + \frac{\rho}{2}\right)|A'||B'|$$

for each $i = 1, \ldots, s$ and

$$|\mathbb{J}_0 \cap (A' \times B')| \leq \left(\frac{\beta}{s+\beta} + \frac{\rho}{4s} \right) |A'||B'|.$$

Then for some partition satisfying this property and for $0 \leq i \leq s$ let J_i be the graph formed by the edges in $\mathbb{J}_i$.

Let i be fixed and suppose $w : A \cup B \to [0,1]$ satisfies the conditions of the lemma. Let us also assume ℓ is chosen large enough so that $\ell \geq 8s/\rho^2$. Then we know $W_i(w) = \sum_{ab \in E(J_i)} w(a)w(b) \leq W_i(w_0)$ for some w_0 (which depends on i) with properties (i), (ii), and (iii). Therefore letting $S_A = \{a \in A : w_0(a) = 1\}$ and $S_B = \{b \in B : w_0(b) = 1\}$ we find

$$\begin{aligned} W_i(w) \leq W_i(w_0) &\leq \sum_{ab \in E(J_i) \cap (S_A \times S_B)} 1 + \sum_{a \in M_A(w_0) \text{ or } b \in M_B(w_0)} w(a)w(b) \\ &\leq |E_{J_i}(S_A, S_B)| + w(A) + w(B). \end{aligned}$$

Now $\ell \geq 8s/\rho^2$ and $w(B) \geq \rho\ell$ imply that $w(A) \leq (\rho/8s)w(A)w(B) < (\rho/4(s+\beta))w(A)w(B)$, and similarly $w(B) \leq (\rho/4(s+\beta))w(A)w(B)$. Therefore

$$W_i(w) \leq \begin{cases} \frac{1}{s+\beta}(1+\rho)w(A)w(B) & \text{if } 1 \leq i \leq s \\ \left(\frac{\beta}{s+\beta} + \frac{\rho}{2s} \right) w(A)w(B) & \text{if } i = 0. \end{cases}$$

This proves the upper bound. For the lower bound, it suffices to note that for $i = 1, \ldots, s$,

$$\begin{aligned} \sum_{ab \in E(J_i)} w(a)w(b) &= \sum_{a \in A, b \in B} w(a)w(b) - \sum_{j \neq i} \sum_{ab \in E(J_j)} w(a)w(b) \\ &\geq w(A)w(B) - \left(\frac{s-1}{s+\beta} \right)(1+\rho)w(A)w(B) - \left(\frac{\beta}{s+\beta} + \frac{\rho}{2s} \right) w(A)w(B) \\ &= w(A)w(B) \left[\frac{1}{s+\beta} - \rho \frac{(s-1)}{s+\beta} - \frac{\rho}{2s} \right] \\ &> \left(\frac{1}{s+\beta} - \rho \right) w(A)w(B). \end{aligned}$$

This completes the proof. ∎

Proof of Lemma 6. Let $\rho = \xi'/10$. Let $\ell = \ell(\lfloor \sigma^{-1} \rfloor, \rho)$ be as in Lemma 16, and let $\xi = (\xi')^3/10\ell$ and $k_0 = \lceil 10\ell/(\xi')^2 \rceil$. Suppose an integer $j \leq \lfloor \sigma^{-1} \rfloor$ and a bipartite ξ-regular graph G with bipartition $U \cup V$ and density $d(U,V)$ are given, where $|U| = |V| = k \geq k_0$ and $j\sigma \leq d(U,V) < (j+1)\sigma$. Let $\beta = (d(U,V) - j\sigma)/\sigma$, and let $J_1, \ldots, J_j$ be the subgraphs of the complete bipartite graph with bipartition $A \cup B$ guaranteed by Lemma 16, where $|A| = |B| = \ell$.

The algorithm is as follows. Since ℓ does not depend on k we first find the graphs $J_1,\dots,J_j$ by brute force. Then we partition U and V in an arbitrary way into subsets $U=U_0\cup\bigcup_{a\in A}U_a$ and $V=V_0\cup\bigcup_{b\in B}V_b$ such that $|U_a|=|V_b|=\lfloor k/\ell\rfloor$ for each $a\in A$, $b\in B$. We then define the subgraphs $G^1,\dots,G^j$ of G as follows. We let $V(G^i)=U\cup V$ for each i, and $E(G^i)=\{uv\in E(G): u\in U_a, v\in V_b \text{ for some } ab\in E(J_i)\}$.

We now check that each G^i is ξ'-regular of density $\sigma\pm\xi'$. Let $U'\subset U$ and $V'\subset V$ be given where $|U'|\ge\xi'|U|$, $|V'|\ge\xi'|V|$. For each $a\in A$, $b\in B$ set $w(a)=|U'\cap U_a|/|U_a|$ and $w(b)=|V'\cap V_b|/|V_b|$. Then the function w satisfies the conditions in Lemma 16, since $w(A)=\sum_{a\in A}w(a)=|U'\setminus U_0|/\lfloor k/\ell\rfloor>(1-\xi'/10)|U'|/\lfloor k/\ell\rfloor>\rho\ell$, and similarly $w(B)>\rho\ell$. Then

$$d_{G^i}(U',V')=\frac{|E_{G^i}(U',V')|}{|U'||V'|}=\sum_{ab\in E(J_i)}\frac{|E_G(U_a\cap U',V_b\cap V')|}{|U'||V'|}$$

$$=\sum_{ab\in E(J_i)}\frac{|E_G(U_a\cap U',V_b\cap V')|}{|U_a\cap U'||V_b\cap V'|}w(a)w(b)\frac{|U_a||V_b|}{|U'||V'|}. \tag{4}$$

Now for each a, b such that $|U_a\cap U'|\ge\xi k$ and $|V_b\cap V'|\ge\xi k$, we know by ξ-regularity that

$$j\sigma+\lambda-\xi=d(U,V)-\xi<\frac{|E_G(U_a\cap U',V_b\cap V')|}{|U_a\cap U'||V_b\cap V'|}<d(U,V)+\xi=j\sigma+\lambda+\xi.$$

But note also that $\sum\{|E_{G^i}(U_a\cap U',V')|:a\in A,|U_a\cap U'|<\xi k\}<\ell\xi k^2$ and $\sum\{|E_{G^i}(U',V_b\cap V')|:b\in B,|V_b\cap V'|<\xi k\}<\ell\xi k^2$. Therefore from (4) and the result of Lemma 16 we find

$$\begin{aligned}d_{G^i}(U',V')&<\frac{j\sigma+\lambda+\xi}{w(A)w(B)}\sum_{ab\in E(J_i)}w(a)w(b)+\frac{2\ell\xi k^2}{|U'||V'|}\\&\le(j\sigma+\lambda+\xi)\left(\frac{1}{j+\lambda/\sigma}+\rho\right)+\frac{2\ell\xi}{(\xi')^2}\\&\le\sigma+(j\sigma+\lambda)\rho+\frac{\xi\sigma}{j\sigma+\lambda}+\xi\rho+\frac{2\ell\xi}{(\xi')^2}\le\sigma+\xi',\end{aligned}$$

since $\rho=\xi'/10$, $\xi=(\xi')^3/10\ell$, and $\sigma<j\sigma+\lambda<2$.

For the lower bound, note that $\sum_{ab\in E(J_i)}w(a)w(b)\le\sum\{w(a)w(b):ab\in E(J_i),w(a)\ge2\xi\ell,w(b)\ge2\xi\ell\}+4\xi\ell^2$, and moreover that $w(a)\ge2\xi\ell$ implies $|U_a\cap U'|\ge\xi k$ and $w(b)\ge2\xi\ell$ implies $|V_b\cap V'|\ge\xi k$. Note also that $w(A)w(B)|U_a||V_b|>(1-\xi'/5)|U'||V'|$. Therefore again using Lemma 16 we have

$$
\begin{aligned}
d_{G^i}(U',V') \geq & \sum_{ab\in E(J_i),|U_a\cap U'|,|V_b\cap V'|\geq \xi k} \frac{|E_G(U_a\cap U',V_b\cap V')|}{|U_a\cap U'||V_b\cap V'|} w(a)w(b)\frac{|U_a||V_b|}{|U'||V'|} \\
& > (j\sigma+\lambda-\xi)\frac{\lfloor k/\ell\rfloor^2}{|U'||V'|} \sum_{ab\in E(J_i),w(a),w(B)\geq 2\xi\ell} w(a)w(b) \\
& > (j\sigma+\lambda-\xi)\frac{\lfloor k/\ell\rfloor^2}{|U'||V'|}\left[(\frac{1}{j+\lambda/\sigma}-\rho)w(A)w(B)-4\xi\ell^2\right] \\
& > (j\sigma+\lambda-\xi)\left[(1-\frac{\xi'}{5})(\frac{1}{j+\lambda/\sigma}-\rho)-\frac{4\xi\ell^2}{(\xi'\ell)^2}\right] \\
& = \sigma-\xi\left[(\frac{1}{j+\lambda/\sigma}-\rho)(1-\frac{\xi'}{5})-\frac{4\xi'}{10\ell}\right]+ \\
& \qquad (j\sigma+\lambda)\left[\frac{\rho\xi'}{5}-\rho-\frac{\xi'}{5(j+\lambda/\sigma)}-\frac{4\xi'}{10\ell}\right] \\
& > \sigma-\xi'.
\end{aligned}
$$

∎

8. Proof of Lemma 8

Given H_0, α, ϵ', and s_0 as in the statement of the lemma, we make the following definitions. We let

$\epsilon = \min\{\alpha/5, \epsilon'\}$, and
$S = S_0(\epsilon, K(\alpha/5, H_0), \max\{s_0, \epsilon^{-1}\})$ (see Lemmas 4 and 3).

Let G be a graph with n vertices. We begin with the construction of the graph $\bar{G}$. Let ψ^* be a maximum fractional H_0-packing in G. Our plan is to apply Lemma 3 to G, where the vertex partition is such that most of the value $|\psi^*|$ of ψ^* comes from crossing copies of H_0. First we apply Lemma 4 to G with parameters $\alpha/5$ and H_0 to obtain a partition $Q = (V_i)_{i=1}^q$ of $V(G)$ such that $q \leq K(\alpha/5, H_0)$ and

$$\sum_{H\in\mathcal{C}(G,Q,H_0)} \psi^*(H) \geq (1-\alpha/5)|\psi^*|. \tag{5}$$

Next, we apply Lemma 3 with parameters ϵ, q, and $\max\{s_0, 1/\epsilon\}$ to G with partition Q. Let $P = (C_i)_{i=0}^s$ be the partition of $V(G)$ given by Lemma 3, so $s \leq S_0(\epsilon, q, \max\{s_0, \epsilon^{-1}\}) \leq S$. Let m denote the size of the vertex classes C_i, $1 \leq i \leq s$. For each pair (C_i, C_l) for which the bipartite subgraph $G[C_i, C_l]$ of G induced by the classes C_i and C_l is ϵ-regular, with some density d_{il}, we let

$J(i,l)=G[C_i,C_l]$. If $G[C_i,C_l]$ is not ϵ-regular, we let $J(i,l)$ be the bipartite graph with vertex classes C_i and C_l with no edges, and we set $d_{il}=0$.

We let $\bar{G}$ be the graph with vertex set $C_1\cup\ldots\cup C_s$ and edge set $\{e\in E(G): e\in E(J(i,l))$ for some $i,l\}$. Then Properties (i) and (ii) in Lemma 8 hold for $\bar{G}$ by construction. To check Property (iii), note first that by Lemma 3 there are at most ϵn^2 edges of G incident to vertices in the exceptional class C_0, at most $\epsilon n^2/2$ contained in irregular pairs, and at most $n^2/2s\le n^2/2s_0$ inside partition classes. Therefore we have

$$\begin{aligned}|E(\bar{G})|-|E(\bar{G})| &\le \epsilon n^2+\epsilon n^2/2+n^2/2s\\ &< 3\epsilon n^2\le 3\epsilon' n^2,\end{aligned}$$

since $s\ge 1/\epsilon$ from our application of Lemma 3. This completes the definition of $\bar{G}$.

Now we turn to Assertion (iv). Using (5) we have

$$\begin{aligned}\sum_{H\in\mathcal{C}(\bar{G},P,H_0)}\psi^*(H) &\ge (1-\alpha/5)|\psi^*|-\sum_{H\in\mathcal{C}(G,P,H_0)\setminus H\in\mathcal{C}(\bar{G},P,H_0)}\psi^*(H)\\ &\ge (1-\alpha/5)|\psi^*|-(|E(G)|-|E(\bar{G})|)\\ &\ge |\psi^*|-\alpha/5|\psi^*|-3\epsilon n^2>|\psi^*|-7\alpha n^2/10,\end{aligned}$$

where the last inequality follows since $|\psi^*|\le\binom{n}{2}<n^2/2$ and $3\epsilon<3\alpha/5$.

For $H\in\mathcal{C}(\bar{G},P,H_0)$, we define the *projection* $\pi(H)\in\binom{G_0}{H_0}$ as follows. Let $V(H)=\{w_1\ldots,w_h\}$ where $w_u\in C_{i_u}$ for $1\le u\le h$. Then we let $\pi(H)$ be the copy of H_0 in G_0 with vertex set $\{c_{i_1},\ldots,c_{i_h}\}$ and edge set $\{c_{i_u}c_{i_v}: w_uw_v\in E(H)\cap E(J(i_u,i_v))\}$.

To prove (iv), we shall exhibit a fractional H_0-packing ψ_0 of G_0 such that $m^2|\psi_0|\ge\nu^*_{H_0}(G)-\alpha n^2$. For $H'\in\binom{G_0}{H_0}$, we define

$$\psi_0(H')=\frac{1}{m^2}\sum_{H\in\mathcal{C}(\bar{G},P,H_0),\pi(H)=H'}\psi^*(H).$$

To see that ψ_0 is a fractional H_0-packing of G_0, let $e_0=c_ic_l$ be any edge of G_0. Then

$$\begin{aligned}\sum_{H'\ni e_0}\psi_0(H') &= \frac{1}{m^2}\sum_{H'\ni e_0}\ \sum_{H\in\mathcal{C}(\bar{G},P,H_0):\pi(H)=H'}\psi^*(H)\\ &=\frac{1}{m^2}\sum_{e\in E(J_t(i,l))}\ \sum_{H\in\mathcal{C}(\bar{G},P,H_0):H\ni e}\psi^*(H)\\ &\le\frac{1}{m^2}|E(J(i,l))|\le d_{il}.\end{aligned}$$

Therefore ψ_0 is a valid fractional H_0-packing of G_0.

Furthermore we have

$$\begin{aligned}|\psi_0| &= \frac{1}{m^2}\sum_{H'\in\binom{G_0}{H_0}}\ \sum_{H\in\mathcal{C}(\bar{G},P,H_0):\pi(H)=H'}\psi^*(H)\\ &= \frac{1}{m^2}\sum_{H\in\mathcal{C}(\bar{G},P,H_0)}\psi^*(H)\\ &\geq \frac{1}{m^2}\left(|\psi^*|-7\alpha n^2/10\right).\end{aligned}$$

This implies that $m^2|\psi_0|\geq\nu^*_{H_0}(G)-7\alpha n^2/10>\nu^*_{H_0}(G)-\alpha n^2$. This completes the proof of (iv).

The above proof gives an efficient algorithm for finding $\bar{G}$ and G_0. For an explicit outline, see the algorithm in Section 4. ∎

9. Proof of Lemma 7

We shall deduce Lemma 7 from a more general result about fractional packings in weighted hypergraphs. For a vertex-weighted hypergraph $\mathcal{H}$ in which every vertex $v\in V(\mathcal{H})$ receives a weight $w(v)$, we say that a function $\phi:\mathcal{H}\to[0,1]$ is a *fractional packing* of $\mathcal{H}$ if $\sum_{E\ni v}\phi(E)\leq w(v)$ for every $v\in V(\mathcal{H})$. We say that a fractional packing is τ-*bounded* for some real number τ if for each $E\in\mathcal{H}$, either $\phi(E)=0$ or $\phi(E)\geq\tau$.

We shall need the following theorems from [12] (see also [2]).

Theorem 17. *Let $X_1,\dots,X_n$ be independent random variables where*

$$X_i=\begin{cases}1 & \text{with probability } p_i\\ 0 & \text{with probability } 1-p_i.\end{cases}$$

Let $X=\sum_{i=1}^n X_i$ and $\lambda=\mathbb{E}[X]=\sum_{i=1}^n p_i$. Then

(a) If $\epsilon<3/2$ then $\mathbb{P}[|X-\lambda|\geq\epsilon\lambda]<2\exp(-\epsilon^2\lambda/3)$.
(b) If $x\geq 7\lambda$ then $\mathbb{P}[X\geq x]\leq\exp(-x)$.

Theorem 18. *Let $\eta>0$ and $r\in\mathbb{N}$ be given. Then there exists $k_0=k_0(r,\eta)$ such that the following holds.*

Let $\mathcal{H}$ be any r-uniform vertex-weighted hypergraph where the weight $w(v)$ of each vertex v satisfies $0\leq w(v)\leq 1$, and let ϕ be any fractional packing of $\mathcal{H}$ such that $\phi(E)<1/k_0$ for every $E\in\mathcal{H}$. Then there exists *a $1/k_0$-bounded fractional packing $\bar{\phi}$ of $\mathcal{H}$ such that $|\bar{\phi}| \geq |\phi| - \eta n$, where $|V(\mathcal{H})| = n$.*

Proof. Clearly we may assume that $|\phi| > \eta n$. For a vertex v we let $\phi(v) = \sum_{E \ni v} \phi(E)$. Then note that $|\phi| = (1/r) \sum_{v \in V} \phi(v)$, where we let $V = V(\mathcal{H})$.

Let k be large enough such that the following hold:

(i) $k^{-1/2} \leq \eta|\phi|/10n$ (note that $k \geq 100/\eta^4$ would do),
(ii) $2\exp(-\eta^2\sqrt{k}/300) < 1/k^2$,
(iii) $400/k^2 < \eta$,
(iv) $\sum_{x \geq 7\sqrt{k}} (x+1)\exp(-x) < 1/k$.

Then we set $k_0 = \lceil k(1+\eta/10) \rceil$.

We begin by removing from $\mathcal{H}$ vertices that have small values of $\phi(v)$. For a fractional packing f of a hypergraph $\mathcal{J}$ we will say that a vertex $v \in V(\mathcal{J})$ is *f-small* if $f(v) < \eta|\phi|/10n$. We shall remove small vertices one by one from $\mathcal{H}$ as follows. Let $\mathcal{H}_0 = \mathcal{H}$ and $\phi_0 = \phi$. Now for $i \geq 0$, if there are no ϕ_i-small vertices in $V(\mathcal{H}_i)$ then set $\phi' = \phi_i$, $V' = V(\mathcal{H}_i)$, $\mathcal{H}' = \mathcal{H}_i$ and stop. Otherwise let x be a ϕ_i-small vertex. Then set $V(\mathcal{H}_{i+1}) = V(\mathcal{H}_i) \setminus \{x\}$, let $\mathcal{H}_{i+1} = \mathcal{H}_i \setminus \{E \in \mathcal{H}_i : v \in E\}$ and define ϕ_{i+1} on $\mathcal{H}_{i+1}$ by $\phi_{i+1}(E) = \phi_i(E)$ for each $E \in \mathcal{H}_{i+1}$.

When this process is completed we have that $\mathcal{H}'$ and ϕ' satisfy the following properties.

(6) $|\phi'| \geq |\phi| - \sum_{v \in V \setminus V'} \phi(v) \geq |\phi| - n\eta|\phi|/10n \geq |\phi|(1-\eta/10)$,
(7) $\phi'(v) \geq \eta|\phi|/10n$ for every $v \in V'$, so in particular by (i) we have $\phi'(v) \geq 1/\sqrt{k}$ for every $v \in V'$.

Now we let $\tilde{\mathcal{H}}$ be a random subset of $\mathcal{H}'$ where each $E \in \mathcal{H}'$ is chosen randomly and independently with probability $p_E = \phi'(E)k < 1$. Let $\tilde{V} = V' = V(\tilde{\mathcal{H}})$. Now for each vertex $v \in \tilde{V}$ we have $\mathbb{E}[d_{\tilde{\mathcal{H}}}(v)] = \sum_{E \ni v} k\phi'(E) = k\phi'(v)$, where we use $d_{\tilde{\mathcal{H}}}(v)$ to denote the number of edges of $\tilde{\mathcal{H}}$ that contain v. Therefore by Theorem 17(a) we have

$$\begin{aligned} \mathbb{P}[|d_{\tilde{\mathcal{H}}}(v) - k\phi'(v)| > \eta/10k\phi'(v)] &< 2\exp(-\eta^2 k\phi'(v)/300) \\ &< 2\exp(-\eta^2\sqrt{k}/300) < 1/k^2, \end{aligned} \tag{8}$$

where the last two inequalities follow from (7) and (ii) respectively.

We shall call a vertex v *bad* if $d_{\tilde{\mathcal{H}}}(v) > (1+\eta/10)k\phi'(v)$ and *thin* if $d_{\tilde{\mathcal{H}}}(v) < (1-\eta/10)k\phi'(v)$. Let B and T denote the sets of bad and thin vertices respectively. Then from (8) we find $\mathbb{E}[|B|] < n/k^2$ and $\mathbb{E}[|T|] < n/k^2$.

Next we check that the number of edges incident to bad vertices is small. For a vertex $v \in \tilde{V}$ we define

$$M_v = \begin{cases} d_{\tilde{\mathcal{H}}}(v) & \text{if } (1+\eta/10)k\phi'(v) \leq d_{\tilde{\mathcal{H}}}(v) < 7\phi'(v)k \\ 0 & \text{otherwise.} \end{cases}$$

Let $M = \sum_{v\in\tilde{V}} M_v$. Then note that M gives an upper bound on the number of edges of $\tilde{\mathcal{H}}$ incident to "moderately bad" vertices, that is, bad vertices with degree at most $7k\phi'(v)$. Then we have

$$\begin{aligned}\mathbb{E}[M] = \sum_{v\in\tilde{V}} \mathbb{E}[M_v] &< 2\sum_{v\in\tilde{V}} d_{\tilde{\mathcal{H}}}(v)\exp(-\eta^2\phi'(v)k/300)\\ &< (2/k^2)\sum_{v\in\tilde{V}} d_{\tilde{\mathcal{H}}}(v)\\ &\le (14/k)\sum_{v\in\tilde{V}}\phi'(v) < 14n/k, \qquad (9)\end{aligned}$$

where the first inequality follows from (8), the second from (ii) and the third from the definition of M_v.

Now define

$$R_v = \begin{cases} d_{\tilde{\mathcal{H}}}(v) & \text{if } d_{\tilde{\mathcal{H}}}(v) \ge 7\phi'(v)k \\ 0 & \text{otherwise.}\end{cases}$$

We let $R=\sum_{v\in\tilde{V}} R_v$, so R gives an upper bound on the number of edges incident to "really bad" vertices v with degree more than $7k\phi'(v)$. Since from (7) we have $7k\phi'(v)\ge 7\sqrt{k}$, from (b) we find

$$\mathbb{E}[R] = \sum_{v\in\tilde{V}}\mathbb{E}[R_v] < \sum_{v\in\tilde{V}}\sum_{x\ge 7\sqrt{k}}(x+1)\exp(-x) < n/k, \qquad (10)$$

where the last inequality follows from (iv). Therefore from (9) and (10) the number m_B of edges incident with bad vertices satisfies $\mathbb{E}[m_B] < 15n/k$. Then since also $\mathbb{E}[|T|] < n/k^2$, there exists some $\tilde{\mathcal{H}}_0$ such that $m_B \le 45n/k$ and $|T| \le 3n/k^2$.

Let $\tilde{\mathcal{H}}_1$ be the hypergraph formed by removing all edges from $\tilde{\mathcal{H}}_0$ that are incident to bad vertices. Then note that by construction $\tilde{\mathcal{H}}_1$ has the property that $d_{\tilde{\mathcal{H}}_1}(v) \le \phi'(v)k(1+\eta/10)$ for every vertex v. We define $\bar{\phi}$ by setting

$$\bar{\phi}(E) = \begin{cases} 1/k(1+\eta/10) & \text{if } E\in\tilde{\mathcal{H}}_1 \\ 0 & \text{otherwise.}\end{cases}$$

Then $\bar{\phi}$ is a fractional packing of $\mathcal{H}$ since for $v\in V$ we have $\sum_{E\ni v}\bar{\phi}(E) = d_{\tilde{\mathcal{H}}_1}(v)/k(1+\eta/10) \le \phi'(v) \le w(v)$. Also

$$|\bar{\phi}| \ge \frac{1}{k(1+\eta/10)}\left[\frac{1}{r}\sum_{v\in\tilde{V}} d_{\tilde{\mathcal{H}}_1}(v)\right]$$

$$
\begin{aligned}
&= \frac{1}{k(1+\eta/10)}\left[\frac{1}{r}\sum_{v\in\tilde V} d_{\tilde{\mathcal{H}}_0}(v) - m_B\right]\\
&\geq \frac{1}{k(1+\eta/10)}\left[\frac{1}{r}\sum_{v\in\tilde V\setminus T} d_{\tilde{\mathcal{H}}_0}(v) - m_B\right]\\
&\geq \frac{1}{k(1+\eta/10)}\left[\frac{1}{r}(1-\eta/10)\sum_{v\in\tilde V\setminus T} \phi'(v)k - m_B\right]\\
&\geq \frac{1}{k(1+\eta/10)}\left[\frac{(1-\eta/10)k}{r}\sum_{v\in V'} \phi'(v) - \frac{|T|}{r}(1-\eta/10)k - m_B\right]\\
&\geq \frac{1-\eta/10)}{(1+\eta/10)}|\phi'| - \frac{3n}{rk^2} - \frac{45n}{k^2(1+\eta/10)}\\
&\geq |\phi'| - \frac{\eta|\phi'|}{5} - \frac{3n}{rk^2} - \frac{45n}{k^2} \geq |\phi'| - \eta n/2,
\end{aligned}
$$

where the last inequality follows from (iii). Therefore $|\bar\phi| \geq |\phi| - \eta n$ by (6). Finally, since $k_0 = \lceil k(1+\eta/10)\rceil$ we see that $\bar\phi$ is $1/k_0$-bounded. ∎

We are now ready to prove Lemma 7.

Proof of Lemma 7. Given H_0 and η, we let $r = |E(H_0|$, and we let $\tau = 1/k_0(r,\eta)$ where k_0 is defined as in Theorem 18. Let G be an edge-weighted graph with n vertices, and let ψ^* be a maximum fractional H_0-packing of G. We define an r-uniform vertex-weighted hypergraph $\mathcal{H}$ as follows. The vertex set of $\mathcal{H}$ is $V(\mathcal{H}) = \{v_e : e \in E(G)\}$, where the weight $w(v_e)$ of a vertex v_e is the weight in G of the corresponding edge e. Then note that $m = |V(\mathcal{H})| = |E(G)| < n^2$. A set of r vertices of $\mathcal{H}$ form an edge of $\mathcal{H}$ if and only if the corresponding r edges form a copy of H_0 in G. Then ψ^* corresponds to a fractional packing ϕ^* of $\mathcal{H}$, such that $|\phi^*| = \nu^*_{H_0}(G)$.

First we modify $\mathcal{H}$ by removing edges E for which $\phi^*(E) \geq \tau$. Let $\mathcal{E}_0 = \{E \in \mathcal{H} : \phi^*(E) \geq \tau\}$. Then we define a new vertex-weighted hypergraph $\mathcal{H}'$ where

$V(\mathcal{H}') = V(\mathcal{H})$,
$w_{\mathcal{H}'}(v) = w_{\mathcal{H}}(v) - \sum_{E\ni v, E\in\mathcal{E}_0} \phi^*(E)$ for each $v \in V(\mathcal{H}')$,
$\mathcal{H}' = \mathcal{H} \setminus \mathcal{E}_0$.

We also define a fractional packing ϕ' on $\mathcal{H}'$ by $\phi'(E) = \phi^*(E)$ for each $E \in \mathcal{H}'$. Then

$$|\phi'| = |\phi^*| - \sum_{E\in\mathcal{E}_0} \phi^*(E). \tag{11}$$

Note then that ϕ' is in fact a fractional packing since for $v \in V(\mathcal{H}')$ we have

$$\sum_{E \ni v} \phi'(E) = \sum_{E \ni v, E \in \mathcal{H}} \phi^*(E) - \sum_{E \ni v, E \in \mathcal{E}_0} \phi^*(E)$$
$$\leq w_{\mathcal{H}}(v) - \sum_{E \ni v, E \in \mathcal{E}_0} \phi^*(E) = w_{\mathcal{H}'}(v).$$

Also, for every $E \in \mathcal{H}'$ we have $\phi'(E) < \tau$. Then since $\tau = 1/k_0(r, \eta)$, by Theorem 18 there exists a τ-bounded fractional packing $\bar{\phi}$ of $\mathcal{H}'$ such that $|\bar{\phi}| \geq |\phi'| - \eta m$. We therefore define a fractional packing ϕ of $\mathcal{H}$ as follows: we let

$$\phi(E) = \begin{cases} \phi^*(E) & \text{if } E \in \mathcal{E}_0 \\ \bar{\phi}(E) & \text{if } E \in \mathcal{H}'. \end{cases}$$

Then ϕ is τ-bounded by construction. Also

$$|\phi| = \sum_{E \in \mathcal{E}_0} \phi^*(E) + |\bar{\phi}|$$
$$\geq \sum_{E \in \mathcal{E}_0} \phi^*(E) + |\phi'| - \eta m$$

by (11)

$$= |\phi^*| - \eta m$$
$$> \nu^*_{H_0}(G) - \eta n^2.$$

Since ϕ corresponds to a fractional H_0-packing of G, the result follows. ∎

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P. E. Haxell

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ont., Canada N2L 3G1

pehaxell@math.uwaterloo.ca

V. Rödl

Department of Mathematics and Computer Science, Emory University, Atlanta, GA, USA 30332

rodl@mathcs.emory.edu